Answer the following problems and simplify your answers.

1. (18pts) Find the **explicit solution** to the following initial value problem:

$$\begin{cases} \dfrac{dz}{dt} - e^{t+z} = 0 \\ z(0) = \ln 2 \end{cases}$$

Solution: Using separation of variables, we have

$$\int e^{-z}\,dz = \int e^{t}\,dt \implies -e^{-z} = e^{t} + C.$$

Solving for z yields

$$z = -\ln(\tilde{C} - e^{t}), \qquad \tilde{C} = -C.$$

Applying initial conditions, we have

$$\ln 2 = -\ln(\tilde{C} - 1) \implies \tilde{C} = \frac{3}{2}.$$

Then, putting everything together, we have

$$\boxed{z = -\ln\left(\frac{3}{2} - e^{t}\right).}$$

2. (18 pts) Conisder the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ on the interval $\frac{1}{2} \le x \le 1$.

   (a) Find the area of the surface obtained by rotating the curve about the y-axis.

   (b) Set up, **but do not evaluate**, the integral with respect to x to find the area of the surface rotated about $y = -2$.

Solution:

(a) First, we compute

$$y' = \frac{x^2}{2} - \frac{1}{2x^2}.$$

Next, we can compute our length element as

$$\begin{aligned} ds = \sqrt{1 + (y')^2}\, dx &= \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx \\ &= \sqrt{1 + \left(\frac{x^2}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\, dx \\ &= \sqrt{\left(\frac{x^2}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2}\, dx \\ &= \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx \\ &= \frac{x^2}{2} + \frac{1}{2x^2}\, dx. \end{aligned}$$

Since we are rotating the curvecu897 377..032.87..032.87..032.87..032.87..032.87..032.87f 64ce

3. (40 pts) Consider the region R bounded by $y = \frac{1}{2}x^2$ and $y = \sqrt{2x}$.

   (a) Sketch and shade R

respectively. Plugging these values in the washer method formula gives our volume as

$$\boxed{V = \int_0^2 (R^2 - r^2)\,\mathrm{d}x = \int_0^2 \left(\left(2 - \frac{x^2}{2}\right)^2 - (2 - \sqrt{2x})^2 \right) \mathrm{d}x.}$$

iii. The base length of each rectangle is given by the vertical distance in R. In this case, the base

$$b = \sqrt{2x} - \frac{x^2}{2}.$$

Then, the height of the region is $h = 3b$ meaning the area of each rectangle is

$$A(x) = b \cdot h = 3\left(\sqrt{2x} - \frac{x^2}{2}\right)^2.$$

4. (24 pts) Determine whether or not the following sequences converge or diverge. Justify your answer! If the sequence converges, find its limit.

(a) $\dfrac{(-1)^{n+1}n}{n^{3/2}+\sqrt{n}}$ (b) $\ln(2n^2+1) - 2\ln(n+1)$ (c) $1 + 4^n \cdot 3^{2-n}$

Solution:

(a) First, we compute

$$\lim_{n\to\infty} \left|\frac{(-1)^{n+1}n}{n^{3/2}+\sqrt{n}}\right| = \lim_{n\to\infty} \frac{n}{n^{3/2}+\sqrt{n}} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/n} = \frac{0}{1+0} = 0$$

Since the absolute value of the sequence converges to zero,

$$\boxed{\lim_{n\to\infty} \frac{(-1)^{n+1}n}{n^{3/2}+\sqrt{n}} = 0.}$$

Finally, since the limit exists and is finite, the sequence $\boxed{\textit{converges.}}$

(b) Using log rules and continuity, we can compute our limit as

$$\lim_{n\to\infty} \left(\ln(2n^2+1) - 2\ln(n+1)\right) = \lim_{n\to\infty} \ln\left(\frac{2n^2+1}{(n+1)^2}\right) = \ln\left(\lim_{n\to\infty} \frac{2+1/n^2}{(1+1/n)^2}\right) = \boxed{\ln 2.}$$

Since the limit exists and is finite, the sequence $\boxed{\textit{converges}}$.

(c) A little algebra yields

$$1 + 4^n \cdot 3^{2-n} = 1 + 3^2\frac{4^n}{3^n} = 1 + 9\left(\frac{4}{3}\right)^n.$$

The last term in our sequence is geometric with $r = 4/3$. Since $4/3 > 1$,

$$\left(\frac{4}{3}\right)^n \to \infty \quad \text{as } n \to \infty$$

meaning the original sequence $\boxed{\textit{diverges}}$ to infinity.