1. Determine if the series converge or diverge. Be sure to fully justify your answer and state what test that you used.

   (a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3n-1}$

   (b) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{6^{n-1}}$

   (c) (8 points) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n-2n^3}{6n^3+3}\right)^n$

Solution: (a) We apply the divergence test to this series:

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{3n-1} \\ &= \lim_{n\to\infty} \frac{n}{3n-1} \quad \frac{1}{n}\end{aligned}$$

(c) We apply the root test to this series:

$$
\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left(\left| \frac{5n - 2n^3}{6n^3+3} \right|^{n} \right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \left| \frac{5n - 2n^3}{6n^3+3} \right| \\
&= \lim_{n\to\infty} \left| \frac{5n - 2n^3}{6n^3+3} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n^3}} \right| \\
&= \lim_{n\to\infty} \left| \frac{\frac{5}{n^2} - 2}{6 + \frac{3}{n^3}} \right| \\
&= \lim_{n\to\infty} \left| -\frac{2}{6} \right| \\
&= \frac{1}{3} \\
&< 1.
\end{aligned}
$$

Since $\frac{1}{3} < 1$, the series absolutely converges by the root test.

2. Determine the interval of convergence and the radius of convergence for the following power series.

(a) (15 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(b) (15 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n!}$

Solution: (a) By inspection, we see that the center of this power series is $a = 0$. We can apply the ratio test to this series to determine its radius of convergence and interval of convergence:

$$
\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-3)^{n+1}x^{n+1}}{\sqrt{(n+1)+1}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| \\
&= \lim_{n\to\infty} 3|x| \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \\
&= \lim_{n\to\infty} 3|x| \cdot \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} \\
&= \lim_{n\to\infty} 3|x| \cdot \frac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+\frac{2}{n}}} \\
&= 3|x|.
\end{aligned}
$$

For this series to absolutely converge, we require that

$$
\begin{aligned}
3\,|x| < 1 \implies & \quad 1 < 3x < 1 \\
\implies & \quad \frac{1}{3} < x < \frac{1}{3}.
\end{aligned}
$$

From this, we see that the radius of convergence is $R = \frac{1}{3}$ and that the tentative interval of convergence is $I = \quad \frac{1}{3}, \frac{1}{3}$

At this endpoint, the series evaluates to

$$\sum_{n=1}^{\infty} \frac{(-3)^n \left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^{\frac{1}{2}}}$$

This is an alternating series, where the positive portion of the terms are given by $b_n = \frac{1}{(n+1)^{\frac{1}{2}}}$

3. (a) 10 points) Start with the Maclauren Series for $\frac{1}{1 \quad x}$ to nd a power series representation for $\frac{1}{1+2x^2}$. Show all work.
(b) (8 points) Use your answer from part (a) to nd its interval of convergence.

Solution:

(a) $\frac{1}{1+2x^2} = \frac{1}{1 \quad (\quad 2x^2)} \quad) =)$